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GENERATION OF NONLINEAR WAVES ON A VISCOELASTIC COATING IN A TURBULENT BOUNDARY LAYER

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Self-induced excitation of periodic nonlinear waves on a viscoelastic coating interacting with a turbulent boundary layer of an incompressible flow is studied. The response of the flow to multiwave excitation of the coating surface is determined in the approximation of small slopes. A system of equations is obtained for complex amplitudes of multiple harmonics of a slow (divergent) wave resulting from the development of hydroelastic instability on a coating with large losses. It is shown that three-wave resonant relations between the harmonics lead to the development of explosive instability, which is stabilized due to the deformation of the mean (over the wave period) shear flow in the boundary layer. Conditions of soft and hard excitation of divergent waves are determined. Based on the calculations performed, qualitative features of excitation of divergent waves in known experiments are explained.

Introduction. The problem of self-induced excitation (generation) of waves on elastic coatings interacting with fluid flows is of interest because of the search for methods of decreasing hydrodynamic resistance (see, for example, [1–3]). This phenomenon is also important for biomechanical fluid flows [4]. Two basic types of hydroelastic (flow-induced) instability were found within the framework of the linear theory: traveling-wave flutter and divergence [2]. The quasistatic instability (divergence) arises in the homogeneous potential flow past the coating, whereas the appearance of the flutter is conditioned by irreversible transfer of energy from the shear flow in the boundary layer to the surface. A strongly dissipative viscoelastic coating and an ideal (elastic) coating were used in [5, 6] to observe divergence and flutter, respectively, in a flow with a turbulent boundary layer (TBL).

Gad-el-Hak et al. [5] identified divergence as a quasistatic instability with the phase velocity of waves smaller than 5% of the main flow velocity. In this case, nonsinusoidal two-dimensional waves of high amplitude with spinous elevations of the surface appeared as soon as the flow velocity became greater than the critical value. The goal of the present work is to find the mechanism of generation of such waves.

Generation of hydroelastic waves on a bounded elastic plate was numerically simulated by Lucey and Carpenter [7], and the potential flow model without the boundary layer was used. Reutov and Rybushkina [8] obtained the Landau equation for monochromatic waves on elastic and viscoelastic coatings interacting with the TBL. Nonlinear processes of competition of "fast" waves excited during the development of the flutter-type instability were studied in [9]. Due to the low phase velocity of slow (divergent) waves, conditions for synchronization of the phases of multiple harmonics arise, which may lead to the formation of nonlinear waves. In the present paper, the quasilinear theory of TBL interaction with a wavy surface, which was constructed in [10], is supplemented by taking into account resonant interactions of multiple harmonics of a periodic divergent wave. As in [8–10], the main small parameter is the surface slope. This weakly nonlinear theory allows one to study the generation of hydroelastic waves only for rather small supercritical values.

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Nevertheless, its construction seems to be useful for understanding the role of the TBL in mechanisms of limiting hydroelastic instability.

1. Wave Divergence of a Viscoelastic Coating in the TBL (Linear Problem). We consider TBL interaction with a wave flexure on an elastic coating. The external region of the TBL merges with a uniform potential flow without the pressure gradient. The one-layer coating made of an incompressible viscoelastic material of density ρ_s and shear modulus G has a thickness d. The propagation velocity of plane transverse waves in this material is $c_t = \sqrt{G/\rho_s}$. The surface-flexure period λ is assumed to be small as compared to the scale of TBL expansion. As is shown in [8, 9], the interaction of such a flexure with the TBL may be assumed to be local.

To describe periodic two-dimensional perturbations on the coating surface, we use a model equation that is a modification of the Kármán equation in the theory of weak flexure of thin plates:

$$\hat{K}w - \left[\frac{s}{\lambda}\int_0^\lambda (w_{x'})^2\,dx'\right]w_{xx} = -p. \tag{1}$$

Here $w(x,t)$ is the displacement of the surface level toward the y axis at the point x at the time t, $p(x,t)$ is the surface-pressure perturbation, $\hat{K}$ is the linear integrodifferential operator, and $s = Gd/(1-\mu)$ ($\mu \approx 0.5$ is Poisson's ratio for an incompressible viscoelastic layer).

The available experimental data on excitation of hydroelastic waves in the TBL refer to one-layer coatings. In this case, the spectral representation of the operator $\hat{K}$ [for excitations of the form $\exp(ikx - i\omega t)$] may be written as follows [11]:

$$\hat{K}(\Omega,\alpha) = \frac{\rho_s c_t^2}{d}\,[\bar{m}(\alpha^2\bar{c}_0^2 - \Omega^2) - i\gamma_t b_0\Omega]. \tag{2}$$

Here $\alpha = kd$ is the dimensionless wavenumber, $\Omega = \omega d/c_t$ is the dimensionless cyclic frequency, and γ_t is the dimensionless parameter of losses; $\bar{m}$, $\bar{c}_0$, and b_0 depend only on α. In the case of a strongly dissipative coating considered below, formula (2) is obtained rigorously (in the asymptotic sense) by expansion in terms of the parameter $1/(\alpha\gamma_t) \ll 1$.

For $\bar{m} =$ const, $b_0 =$ const, and an appropriate definition of $\bar{c}_0$, relation (2) describes the complex elasticity of a thin plate (or membrane) with a dimensionless surface density $\bar{m}$ and the coefficient of losses $\gamma_t b_0$. In the range of wavenumbers $\alpha \geqslant 1$ (typical of the experiments of [5]), the changes in $\bar{m}$ and b_0 are comparatively small, and their rapid increase begins at small α (there is a singular increase as $\alpha \to 0$).

Thus, in the linear problem with definition of $\hat{K}$ in the form of (2), Eq. (1) describes the surface deformation of an actual viscoelastic layer. In passing to the nonlinear equation (1), the viscoelastic layer is considered as a thin free plate of thickness d whose parameters change in accordance with (2).

Pressure perturbations in the form of a traveling sinusoidal wave, which arise in the TBL with surface flexure, were found in the linear approximation in [11] on the basis of the flow model with vortex viscosity (see also [8]). A quasipotential approximation for the complex elasticity of the flow was proposed. For $\alpha > 0$, this approximation reduces to a relation between the complex amplitudes of pressure $p_{\omega k}$ and flexure $w_{\omega k}$ of the form

$$p_{\omega k} \simeq k\rho_0 U^2\Big[-\Big(\frac{\omega}{kU} - f\Big)^2 + \delta Z^{(0)}\Big]w_{\omega k}, \tag{3}$$

where $f < 1$ is the parameter of reduction of static elasticity of the potential flow, $\delta Z^{(0)}$ is the resistive component of elasticity, and U is the velocity of the uniform flow. Reutov and Rybushkina [11] gave an analytical approximation of the dependence of f on the dimensionless wavenumber $\bar{k} = k\delta_*$ and local Reynolds number $\mathrm{Re} = U\delta_*/\nu_0$ (δ_* is the TBL displacement thickness and ν_0 is the kinematic viscosity of the liquid).

Ignoring the nonlinear term in (1) and using (2) and (3), we obtain the dispersion relation for hydroelastic waves in the form [11]

$$\alpha\bar{m}(\Omega^2 - \alpha^2\bar{c}_0^2) + q(\Omega - \alpha f V)^2 + i\gamma_t b_0\alpha\Omega - qV^2\alpha^2\delta Z^{(0)} = 0, \tag{4}$$

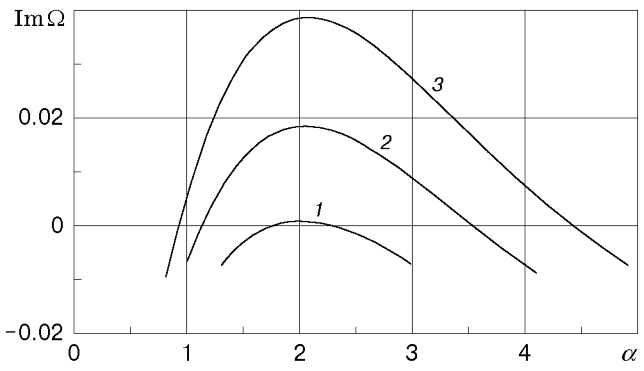

Fig. 1

where $V = U/c_t$ is the dimensionless flow velocity and $q = \rho_0/\rho_s$. For a viscoelastic coating, we have $\gamma_t b_0 = 6$–600, which allows us to seek the solution of Eq. (4) in the form of expansion in $\varepsilon_\alpha = 1/(\gamma_t b_0) \ll 1$ [11]. With accuracy to terms of order ε_α, we obtain the relation for the frequency of weakly decaying (or weakly growing) waves

$$\Omega = i\alpha\varepsilon_\alpha(qf^2V^2 - \alpha\bar{m}\bar{c}_0^2 - qV^2\delta Z_0^{(0)}), \tag{5}$$

where $\delta Z_0^{(0)} = \delta Z^{(0)}\Big|_{\Omega=0}$. Note that Eqs. (4) and (5) contain implicitly the parameters $\mathrm{Re}_t = c_t\delta_*/\nu_0$ and d/δ_* [11].

Figure 1 shows the dependence of the instability growth rate $\mathrm{Im}\,\Omega$ on α for different flow velocities (curves 1–3 correspond to $V = 5.6$, 6.1, and 6.6, respectively). Hereinafter, we give the calculation results for the flow–coating system with the parameters $q = 1$, $\gamma_t = 15$, $\mathrm{Re}_t = 350$, and $d/\delta_* = 0.78$ (see [11]). Similar results were obtained for different values of γ_t, Re_t, and d/δ_*.

As follows from Fig. 1, the instability arises when the flow velocity passes through the critical value $V_{\mathrm{cr}} \simeq 5.6$. A wave with a wavenumber $\alpha_{\mathrm{cr}} \simeq 2$ (wavelength $\lambda_{\mathrm{cr}} = 2\pi d/\alpha_{\mathrm{cr}}$) is excited at the boundary of instability origination. The phase velocity of growing waves is small as compared to the flow velocity, which allows us to consider this instability as wave divergence [2].

Note that the resistive component of the TBL response $\delta Z^{(0)}$ has a weak effect on the behavior of the curves plotted in Fig. 1 and also on the magnitude of the critical flow velocity V_{cr}. In fact, the instability arises when the static stability of the flow exceeds the static stability of the coating with arbitrary losses in the coating. This behavior of the flow–coating system is a consequence of the dominating contribution of the dissipative component [term $i\gamma_t b_0\Omega$ in (2)] to the dynamic elasticity of the coating. In this case, the influence of reactive and resistive components of elasticity of the flow and coating on wave propagation is oppositely different for coatings with low and high losses. In particular, the real part of $\delta Z^{(0)}$, which determines the wave-frequency shift in the case of low losses in the coating, makes a contribution to the decay (amplification) of waves for a strongly dissipative coating. Obviously, similar changes should also occur for nonlinear elasticity of the flow and coating.

2. Derivation of a Closed System of Equations for the Amplitudes of Surface-Flexure Harmonics. To determine the nonlinear response of the TBL to the wave flexure of the surface, Reutov [10] used a quasilinear approximation, where the main manifestation of nonlinearity is related to the deformation of the mean (over the wave period) flow in the TBL. Within the framework of this approximation, the Landau equation was obtained [8]. As was noted in [10], the contribution of the second harmonic of hydrodynamic fields to the nonlinear response is small under the condition that this harmonic either is absent in the surface flexure or has an order $k\tilde{a} \ll 1$ as compared to the first harmonic (k and $\tilde{a}$ are the characteristic values of the wavenumber and surface-deflection amplitude). Nevertheless, this condition may be violated in the presence of the resonance of phase velocities of the first and second harmonics. In the case of divergent waves whose phase velocity is small for $\varepsilon_\alpha \ll 1$, the resonance conditions are satisfied not only for the second harmonic but also for higher multiple harmonics. This may lead to generation of divergent waves of a significantly nonsinusoidal shape.

In deriving the equation for the amplitudes of harmonics of a nonlinear wave, one should take into account the resonant interaction between the harmonics in the quasilinear model of the TBL response. As in [8–10], we use the approximation of weak nonlinearity, assuming the surface slope to be small: $k\tilde{a} \ll 1$. We pass in (1) to the dimensionless time $t_1 = c_t t/d$, coordinate $x_1 = x/d$, and surface elevation $\bar{w} = w/d$. We assume in (1) that $p = p_{\text{q.p.}} + p_{\text{n.l.}}$, where $p_{\text{q.p.}}$ is the quasipotential linear response of the flow to the wave flexure of the surface determined by formula (3) in the spectral representation and $p_{\text{n.l.}}$ is the nonlinear component of the response.

To pass from the spectral relations (2) and (3) to space–time expressions, we use the formal substitutions $\Omega \to \hat{\Omega} = i\partial/\partial t_1$ and $\alpha \to \hat{\alpha} = -i\partial/\partial x_1$. First, we obtain the equation of excitation of multiple harmonics by the "nonlinear force" $p_{\text{n.l.}}$, considering it as a given function of x_1 and t_1. The surface flexure is represented in the form of a set of harmonics:

$$\bar{w}(x_1, t_1) = \sum_{\alpha} a_\alpha(\varepsilon_\alpha t_1)\, \mathrm{e}^{i\alpha x_1}, \tag{6}$$

where a_α is the normalized complex amplitude of the flexure harmonic with wavenumber α. Since $\bar{w}$ is a real quantity, the condition $a_\alpha = a^*_{-\alpha}$ is satisfied (the asterisk denotes complex conjugation). In the case of multiple harmonics, we have $\alpha = n\alpha_1$, where $n = 1, 2, 3, \ldots$ is the number of the harmonic and α_1 is the wavenumber of the first harmonic (the length of the nonlinear wave is $\lambda = 2\pi d/\alpha_1$).

Substituting (6) into (1) and retaining terms of the order of unity, we obtain the following equation of excitation of harmonics with $\alpha > 0$:

$$\frac{da_\alpha}{dt_1} = \gamma_\alpha a_\alpha - \varepsilon_\alpha \left[\sum_{\beta>0} K^{(1)}_{\alpha\beta} |a_\beta|^2 a_\alpha + \frac{2}{\rho_s c_t^2} \, (p_{\text{n.l.}})_\alpha \right]. \tag{7}$$

Here

$$\gamma_\alpha = \alpha\varepsilon_\alpha (qf^2V^2 - \alpha\bar{m}\bar{c}_0^2 - qV^2\delta Z_0^{(0)}), \quad (p_{\text{n.l.}})_\alpha = \frac{1}{\lambda}\int_0^\lambda p_{\text{n.l.}} \mathrm{e}^{-i\alpha x_1}\, dx_1\Big|_{\hat{\Omega}=0}, \quad K^{(1)}_{\alpha\beta} = \frac{\alpha^2\beta^2}{2(1-\mu)}.$$

We note that the nonlinear forces in this approximation are found with ignored derivatives with respect to t in hydrodynamic equations (quasisteady flow). In passing to (7), the condition of small slopes of the surface was not used in the explicit form, but it is present implicitly as one of the conditions of applicability of the theory of weak flexure of thin plates.

The quasilinear TBL response in the case of a multiwave (multiharmonic) flexure of the surface was determined in [9] on the basis of generalization of the monoharmonic quasilinear theory [8]. The numerical scheme proposed in [9] allows one to calculate the expansion for surface-pressure harmonics of zero frequency, which has the following form:

$$p_k = \left[Z^{(0)}(\bar{k}) + \sum_{k_0>0} Z^{(1)}(\bar{k}, \bar{k}_0) |k_0 \tilde{a}_{k_0}|^2 \right] \tilde{a}_k. \tag{8}$$

Here $\tilde{a}_k = a_\alpha d$ are the nonnormalized complex amplitudes of the flexure harmonics, $Z^{(0)}$ is the complex linear elasticity of the flow [which coincides with the total value of the coefficient at $w_{\omega k}$ in the right part of Eq. (3)], and $Z^{(1)}(\bar{k}, \bar{k}_0)$ are the coefficients of the matrix of mutual nonlinear elasticities of the flow for harmonics with wavenumbers $\bar{k}$ and $\bar{k}_0 = k_0\delta_*$. We note that a relation similar to (8) may also be written for the coating response excited by an external pressure field; instead of $Z^{(1)}$, it will contain the quantities $K^{(1)}_{\alpha\beta}$ from (7) (see [9]).

Three-wave resonant relations between the flexure harmonics are determined by second-order terms of the expansion $p_{\text{n.l.}}$ in the small amplitudes of the harmonics. Their calculation for a TBL with vortex viscosity is related to a substantial complication of the numerical procedure described in [9]. At the same time, the estimates of the characteristic parameters of the TBL and divergent waves show that the penetration depth of the oscillating (with a wave period) flow into the TBL k^{-1} is significantly greater than the thickness of the buffer TBL region $y_b \simeq 30\nu_0/u_*$ (u_* is the dynamic velocity of the TBL) and than the length of decay of vortex perturbations, which is evaluated as $\sqrt{\nu_b/(kU_b)}$ (ν_b and U_b are the values of effective viscosity and flow velocity for $y = y_b$). Note that the layers where the phase velocities of the harmonics of the divergent wave coincide with the flow velocity are located deep in the viscous sublayer.

Taking into account these features of divergent waves, we use the model of the potential flow with a reduced velocity $U \to f_1 U$ ($f_1 < 1$ is the coefficient of reduction of the free-stream velocity) to calculate the second-order terms of the expansion of $p_{\text{n.l.}}$. We assume that $f_1 = U_b/U \simeq 0.5$, which is in agreement with the value of the coefficient f in formula (3), which has a similar meaning. Being written with accuracy to second-order terms, the system of equations and boundary conditions for perturbations of the potential φ and the expression for surface pressure take the following form:

$$\varphi_{xx} + \varphi_{yy} = 0 \quad (y > 0), \qquad f_1 U w_x - \varphi_y = -\varphi_x w_x + w\varphi_{yy}\Big|_{y=0},$$
$$p/\rho_0 = -f_1 U(\varphi_x + w\varphi_{xy}) - (1/2)(\varphi_x)^2 - (1/2)(\varphi_y)^2\Big|_{y=0}. \tag{9}$$

We substitute expression (6) for w written in the initial (nonnormalized) variables into (9). Then, with accuracy to first-order terms, we obtain the expression for potential perturbations:

$$\varphi = -\sum_k \frac{ikf_1U}{|k|}\,\tilde{a}_k\,\mathrm{e}^{ikx-|k|y}.$$

Calculating p with accuracy to second-order terms, we find

$$p_k = \rho_0 f_1^2 U^2\Big(-|k|\tilde{a}_k + \sum_{k_0} S_{kk_0}\tilde{a}_{k_0}\tilde{a}_{k-k_0}\Big), \tag{10}$$

where $S_{kk_0} = (1/2)[-(|k_0| + |k|)|k - k_0| - |kk_0| + k^2 + k_0^2 - kk_0]$. Composing $p_{\text{n.l.}}$ from the second terms in expansions (8) and (10) and passing to dimensionless variables, we can transform Eq. (7) to

$$\frac{da_\alpha}{dt_1} = \gamma_\alpha a_\alpha + \sum_\beta \sigma_{\alpha\beta} a_\beta a_{\alpha-\beta} - \sum_{\beta>0} T_{\alpha\beta}|a_\beta|^2 a_\alpha, \tag{11}$$

where $T_{\alpha\beta} = \varepsilon_\alpha(K_{\alpha\beta}^{(1)} + qV^2\alpha\beta^2 Z_{kk_0}^{(1)})$, $\sigma_{\alpha\beta} = -(1/2)\varepsilon_\alpha q f_1^2 V^2(d^2 S_{kk_0})$, $k = \alpha/d$, and $k_0 = \beta/d$.

System (11) for complex amplitudes of harmonics is known in the theory of nonlinear waves as the equations of resonant and asynchronous (energetic) interaction of waves (see, for example, [12]). The first term in the right part determines the growth rates and linear shifts of the harmonic frequencies, the second one is responsible for the resonant relations between the harmonics, and the third term determines the nonlinear decay and nonlinear shifts of the harmonic frequencies.

It should be noted that the deformation of the mean flow in the TBL leads to high values of the coefficients $Z_{kk_0}^{(1)}$ in expansion (8) (see [9, 10] for more details). Therefore, the contributions of the second- and third-order terms in Eq. (11) may be comparable for small slopes of the surface for which expansions (8) and (10) are applicable. Note that there is no such anomaly of $Z_{kk_0}^{(1)}$ in the case of a purely potential flow over the surface (because of the absence of the mean-flow deformation).

3. Explosive Instability and Hard Excitation of Nonlinear Divergent Waves. A procedure for calculating the coefficients $Z^{(1)}$ in expansion (8) for the TBL with vortex viscosity was developed in [9]. Calculations for typical values of $\bar{k}$ and Re performed on the basis of this procedure showed that the real parts of $Z^{(1)}$ are always positive. Thus, the dissipative cubic nonlinearity in (11) plays a stabilizing role.

To identify the role of the "resonant" terms in (11), we represent the corresponding sum in the following form:

$$\sum_\beta \sigma_{\alpha\beta}a_\beta a_{\alpha-\beta} = \sum_{0<\beta<\alpha/2} \sigma_{\alpha\beta}^{(1)} a_\beta a_{\alpha-\beta} + \sum_{\beta>\alpha} \sigma_{\alpha\beta}^{(2)} a_\beta a_{\beta-\alpha}^* + \sigma_\alpha^{(3)} a_{\alpha/2}^2. \tag{12}$$

Here $\sigma_{\alpha\beta}^{(1)} = \varepsilon_\alpha q f_1^2 V^2 \beta(\alpha-\beta)$, $\sigma_{\alpha\beta}^{(2)} = \varepsilon_\alpha q f_1^2 V^2 \alpha(\beta - \alpha)$, $\sigma_\alpha^{(3)} = (1/8)q\alpha^2 f_1^2 V^2$, and $\alpha = n\alpha_1$ and $\beta = m\alpha_1$ ($n, m = 1, 2, 3, \ldots, N$). Summation in the right part of (12) is performed only for positive wavenumbers, and terms with repeated combinations of amplitudes are eliminated (the last term appears only for even n).

It follows from the above relations for $\sigma^{(1)}_{\alpha\beta}$, $\sigma^{(2)}_{\alpha\beta}$, and $\sigma^{(3)}_{\alpha}$ that these quantities are real and positive. We assume that the wave amplitudes are rather small and the cubic terms in (11) may be ignored. It is easily seen that system (11) with substitution (12) describes the so-called explosive instability of waves [12–15], which is manifested in the asymptotic solutions of the form $a_\alpha \to C_\alpha/(t_{1\infty} - t_1)$ as $t_1 \to t_{1\infty}$ [$t_{1\infty} \sim C_\alpha/a_\alpha(0)$ is the time of "explosion" and $C_\alpha = \text{const}$]. Explosive instability was studied in the theory of plasma waves (see, for example, [13]), where it was mainly associated with three-wave interactions in media close to conservative (interaction of waves with different signs of energy). An example of explosive instability in a dissipative medium is the instability of wave triplets in the boundary layer [14]. An explosive growth of waves due to the pure dissipative resonant relations between them was found for electromagnetic waves in a waveguide with a nonlinear leakage current [15].

As was noted above, the dissipative character of the dynamic response of the coating leads to the fact that reactive nonlinearities of the type of coating elasticity and flow elasticity (determined by the coefficients $K^{(1)}$ and $\operatorname{Re} Z^{(1)}$, respectively) are transformed to dissipative nonlinearities (nonlinear decay) after passing to equations of bound waves (11). The "quadratic" part of the flow response is also transformed to the dissipative from the reactive form. Finally, this leads to explosive instability.

A typical feature of explosive instability is the synchronization of the phases of interacting waves for $t_1 \to t_{1\infty}$ [12, 13]. It is easily seen that, in the case of multiple harmonics, the phase differences $\varphi_2 - 2\varphi_1$, $\varphi_3 - \varphi_2 - \varphi_1$, etc., tend to zero ($\varphi_n = \arg a_n$, where n is the number of the harmonic). Without loss of generality, we may assume that $\varphi_n \to 0$. Thus, in the case of hydroelastic waves, explosive instability should lead to the formation of nonlinear waves with spinous elevations of the surface. Divergent waves of this kind were observed in experiments on a viscoelastic coating [5]. A similar process of explosive interaction of harmonics with the formation of nonlinear electromagnetic waves was observed in a waveguide with a nonlinear leakage current [15].

Cubic decay stabilizes the explosive instability and leads to a steady regime with a finite amplitude [12, 15]. Nonlinear shifts of harmonic frequencies also exert a stabilizing effect (contribution of terms of the order of $\operatorname{Im} Z^{(1)}$). In media with explosive instabilities, hard excitation of waves and the associated hysteresis phenomenon are possible.

To study the transition from soft to hard excitation of waves, we consider the limiting case of real γ_α and $T_{\alpha\beta}$, where a_α are real quantities in the regime of phase synchronization. Let the wavenumber of the first harmonic α_1 be equal to the critical wavenumber α_{cr} (see Fig. 1), the flow velocity be close to the critical value V_{cr}, and the second harmonic decay ($\gamma_2 < 0$). Assuming that $|a_2| \ll |a_1|$, we obtain the following system for real $a_{1,2}$:

$$\frac{da_1}{dt_1} = \sigma_{12} a_1 a_2 - T_{11} a_1^3 + \gamma_1 a_1, \qquad \frac{da_2}{dt_1} = \sigma_2 a_1^2 - T_{21} a_1^2 a_2 - |\gamma_2| a_2. \tag{13}$$

In the steady state ($d/dt_1 = 0$), Eq. (13) becomes

$$|a_1| = \left[\frac{r \pm \sqrt{r^2 + 4\gamma_1 T_{11} T_{21}}}{2T_{11}T_{21}}\right]^{1/2}, \tag{14}$$

where $r = \sigma_{12}\sigma_2 - |\gamma_2| T_{11}$. According to (14), soft excitation is observed for $r < 0$. In this case, the nonlinear decay of the wave a_1 and the linear decay of the wave a_2 suppress the explosive growth. Under the condition $r > 0$, hard excitation is observed in the region $\gamma_1 < 0$ with a hysteresis in the dependence of the steady amplitude on the supercritical value of γ_1. It should be taken into account that, for the "nonzero" equilibrium state (13), which appears for $\gamma_1 = 0$ and $r > 0$, the assumption $|a_2| \ll |a_1|$ is justified only in the degenerate case of small r. Obviously, in the absence of this degeneration, a nonlinear wave may appear already for $\gamma_1 \to 0$; the amplitude of the second and even higher harmonics in this wave is comparable to a_1.

To study the excitation of nonlinear waves without restrictions used in deriving Eq. (13), system (11) was solved numerically. System (11) was written for real variables $\operatorname{Re}(a_\alpha)$ and $\operatorname{Im}(a_\alpha)$ and integrated by the Runge–Kutta method for a finite number of harmonics N. The effect of the choice of the number of

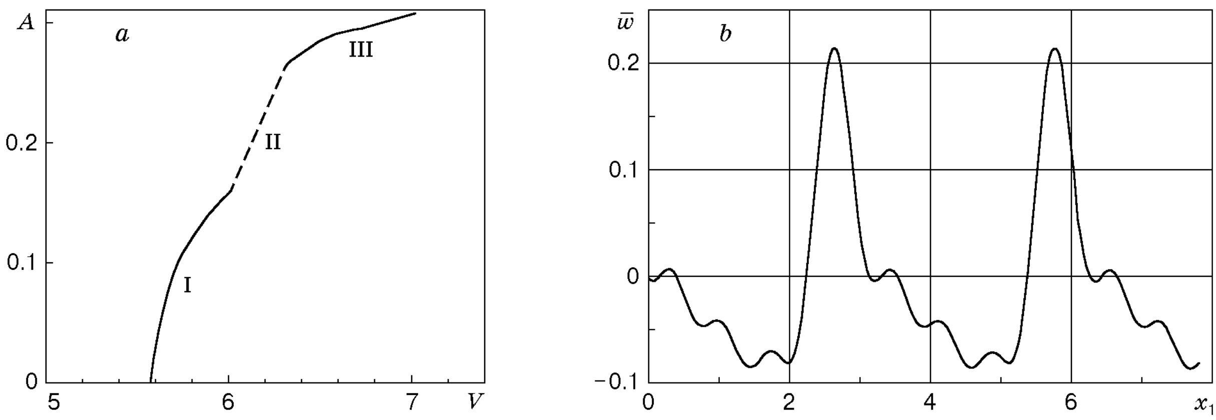

Fig. 2

harmonics N on calculation results was checked. The calculations were performed for the values of the parameters given in Sec. 1.

Figure 2a shows the height of surface elevation in the nonlinear wave $A = w_{\max} - w_{\min}$ as a function of the flow velocity V for the case where the wavenumber of the first harmonic coincides with the critical value $\alpha_1 = \alpha_{\rm cr} \approx 2$ (the calculation was performed for $N = 4$). Regions with a qualitatively different behavior of the solutions are marked by I–III in Fig. 2a. In this case, soft excitation is observed, which is similar to that obtained above in the model with two harmonics. For small supercritical values (region I), the excited waves are close to the sinusoidal shape. In the transitional region II, a rapid growth of multiple harmonics and periodic beatings are observed (there are no steady waves). With further increase in flow velocity (region III), a nonlinear wave is formed; its profile is shown in Fig. 2b. Taking into account the rapid jumplike increase in amplitude A (Fig. 2a), this excitation of nonlinear waves may be called pseudohard.

In the experiments of [5], wave excitation occurred in a different manner. Nonlinear divergent waves with a large height of surface elevation ($A \approx 0.45$) appeared immediately after loss of stability. Reutov and Rybushkina [8] noted that the length of these waves was approximately twice the critical value $\lambda_{\rm cr}$ predicted by the linear theory. Thus, using system (11), we studied the excitation of nonlinear waves with a wavelength $2\lambda_{\rm cr}$ ($\alpha_1 = \alpha_{\rm cr}/2$).

Figure 3a shows the dependence $A(V)$ obtained for $N = 8$. For small supercritical values, when the amplitude of the first harmonic a_1 is small as compared to the amplitude of the second harmonic a_2, the behavior of the dependence $A(V)$ is similar to that in the above-considered case $\alpha_1 = \alpha_{\rm cr}$. Nevertheless, with increase in the supercritical values, an amplitude jump occurs at $V \approx 6$, and a hysteresis in the dependence $A(V)$ appears, which corresponds to hard excitation. The increase in time of the total amplitude $\bar{A} = \Sigma_\alpha |a_\alpha|$ on both sides of the jump is plotted in Fig. 3b (the solid and dashed curves refer to $V = 6.05$ and 5.95, respectively). Figure 3c shows the profile of the nonlinear wave for $V = 6.05$. It follows from the result presented that soft excitation of the wave with a wavelength of $\lambda_{\rm cr}$ leads to the explosive growth of harmonics; this leads to the jumplike appearance of a nonlinear wave with a wavelength of $2\lambda_{\rm cr}$. Note that the value $A \approx 0.45$ for this wave is in agreement with experimental data. Thus, the main qualitative features of wave excitation on a viscoelastic coating in the TBL are a consequence of the explosive character of interaction of multiple harmonics of the nonlinear wave with a period of $2\lambda_{\rm cr}$.

Substitution of $K^{(1)}_{\alpha\beta} = 0$ into the coefficients of Eq. (11) leads to insignificant changes in the calculation results presented above. This means that, for small supercritical values, the main contribution to instability saturation is made by the hydrodynamic nonlinearity related to the mean-flow deformation in the TBL. The dominating role of hydrodynamic nonlinearity is a consequence of the anomaly of numerical values of $Z^{(1)}_{kk_0}$.

Soft excitation of a wave with a wavelength of $\lambda_{\rm cr}$ preceding the jumplike appearance of a nonlinear wave was not observed in the experiments. The reason may be large-scale oscillations of the flow velocity in the boundary layer, which may induce a strongly expressed unsteadiness of the wave field. In addition, the

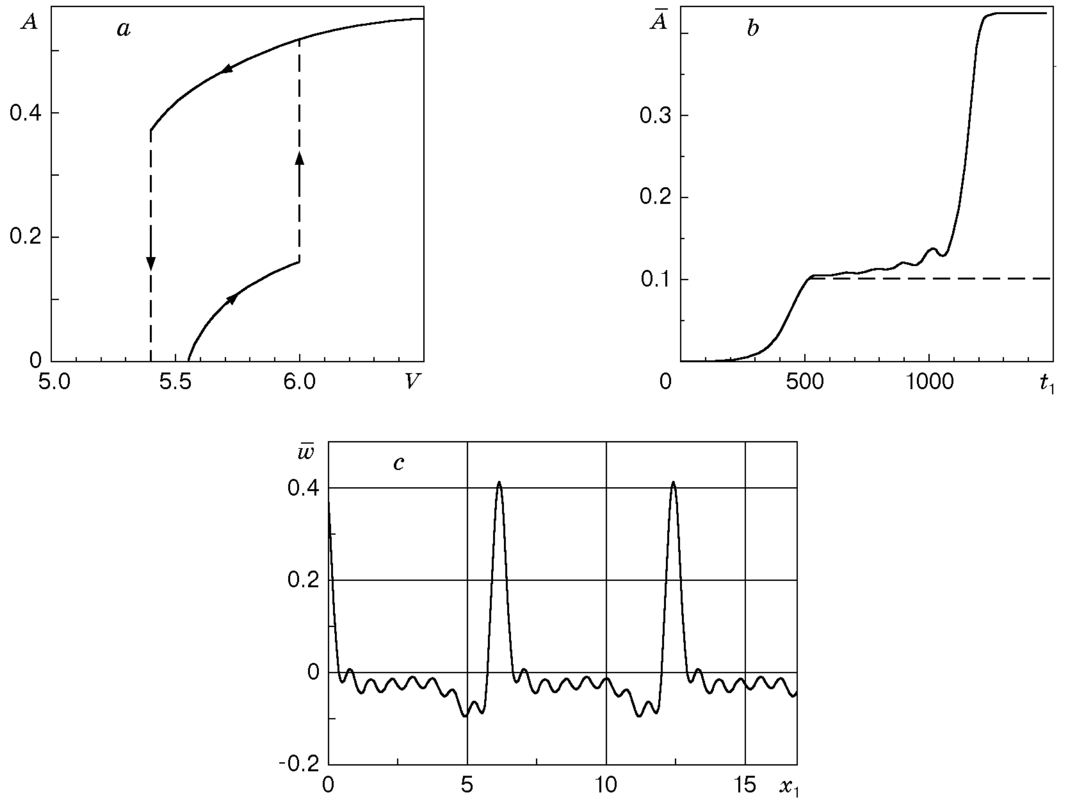

Fig. 3

calculations showed that the interval of soft excitation in Fig. 3a decreases with increasing parameter f_1 (see Sec. 2) and vanishes at $f_1 = 0.71$.

The theoretical profile of the excited wave (see Fig. 3c) has a more nonlinear shape (greater ratio of the wave period to the peak width) than in the experiments; in calculations, the amplitude of the nonlinear wave increases more slowly with increasing V. The reason for this disagreement of the theoretical and experimental data may be, in particular, the insufficient adequacy of the quasipotential approximation used for calculating the coefficients of resonant relations between the harmonics.

Conclusions. The mechanism of excitation of significantly nonlinear divergent waves on a viscoelastic coating in the TBL of an incompressible flow is considered in this work. As a result of resonant interaction of multiple harmonics with low phase velocities, the wave profile becomes significantly nonsinusoidal already for moderate supercritical values. At high losses in the coating, conservative nonlinearities of the type of the flow and coating elasticity in equations of bound waves are transformed to dissipative nonlinearities. The dissipative resonant relation between the harmonics leads to their explosive instability. The main reason for limitation of the explosive growth of waves is the deformation of the mean (over the period of the nonlinear wave) shear flow in the TBL.

Phase synchronization of multiple harmonics related to the development of explosive instability determines the spinous shape of surface-shear waves. From the viewpoint of the general theory of explosive instability, the "nonevolutionary" character with increasing supercritical values should be noted. In this case, the explosive interaction of multiple harmonics leads to the hard excitation of nonlinear waves or stimulates a fast transition to waves of a significantly nonsinusoidal shape (pseudohard excitation).

Because of the rather strong assumptions accepted, the problem considered is, in fact, a model one. Nevertheless, the analysis allowed us to explain the main features of excitation of divergent waves in the experiment: the jumplike origination of nonlinear waves with a typical spinous shape of surface shear and also the excitation of longer waves than those obtained within the framework of the linear theory of stability.

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